Transport properties of clean and disordered superconductors in matrix field theory

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A comprehensive field theory is developed for superconductors with quenched disorder. We first show that the matrix field theory, used previously to describe a disordered Fermi liquid and a disordered itinerant ferromagnet, also has a saddle-point solution that describes a disordered superconductor. A general gap equation is obtained. We then expand about the saddle point to Gaussian order to explicitly obtain the physical correlation functions. The ultrasonic attenuation, number density susceptibility, spin density susceptibility and the electrical conductivity are used as examples. Results in the clean limit and in the disordered case are discussed respectively. This formalism is expected to be a powerful tool to study the quantum phase transitions between the normal metal state and the superconductor state.

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I. INTRODUCTION

The description of disordered many-electron systems is a difficult problem in modern theoretical physics. Landau Fermi-liquid theory[1] and then many-body perturbation theory[2,3] were introduced to deal with this problem. Considerable progress has been made within the framework of the latter. On the other hand, field-theoretic method has also been applied to the many-electron problem, which has certain advantages over the traditional technique. It is relatively easy to include the quenched disorder in the formalism with the help of replica trick. More importantly, it is the natural language to describe any classical or quantum phase transition. It allows for a straightforward application of the renormalization group, implementing an old program of describing the various phases of many-body systems in terms of stable RG fixed points.[4] So far this program has been carried out for clean and disordered Fermi liquids, as well as disordered ferromagnetic metals.[5,6]

In the present paper we will develop a comprehensive field-theoretical method, or matrix field theory,[6] for gapped disordered spin-singlet superconductors. The formalism is expected to be able to describe the quantum phase transition between the normal metal state and the superconductor state. Some similar techniques have been applied to describing spin-triplet, even-parity superconductors, but explicit quantitative expressions for the Gaussian propagators could not be given in the previous paper and thus that description cannot be said to be complete.[7] Here we will completely determine all the soft, or gapless correlation functions for the $S = 0$, spin-singlet case and obtain the corresponding transport properties. The method can be generalized to evaluate other physical systems, like spin-triplet superconductors. Our results for the spin-singlet case coincide with earlier ones obtained by conventional methods. Our field theoretic methods, however, have the advantage that they can be easily generalized to describe quantum phase transitions. For example, in future publications we will use these results to (1) describe a metal–superconductor transition in a dirty metal, without integrating out the soft fermionic degrees of freedom[8] and (2) consider the same superconductor–metal transition from the superconducting side of the transition.[9] This second problem is nontrivial and interesting due to the numerous dangerous irrelevant variables at this phase transition.

The paper is organized as follows. In Sec. II we give a field-theoretic formulation of the problem. In Sec. III we develop the theory to construct a saddle-point solution, obtain the gap equation of superconductivity and expand to the second or Gaussian order about the saddle point. In Sec. IV we show how to calculate the ultrasonic attenuation coefficient, the number and spin density susceptibilities, and the electrical conductivity for both clean and disordered superconductors. In Sec. V we conclude with a general discussion of our results. In Appendix we give some technical points that are used in the paper.

II. MATRIX FIELD THEORY

A. Grassmannian field theory

In general, a system of interacting, disordered electrons will be considered. The partition function of the system is[10]

$$Z = \int D[\bar{\psi}, \psi] \; e^{S[\bar{\psi},\psi]} \quad . \tag{2.1}$$

Here the $\bar{\psi}$ and ψ are Grassmann valued fields, and S is the action including three parts:

$$S = S_0 + S_{\rm int} + S_{\rm dis} \quad , \tag{2.2a}$$

Where, with a $(d+1)$-vector notation $x = (\mathbf{x}, \tau)$ and $\int dx = \int_V d\mathbf{x} \int_0^\beta d\tau$, S_0 describes free electrons with chemical potential μ,

$$S_0 = \int dx \sum_\sigma \; \bar{\psi}_\sigma(x) \left(-\partial_\tau + \frac{\nabla^2}{2m} + \mu \right) \psi_\sigma(x) \quad , \tag{2.2b}$$

$S_{\rm int}$ describes a spin–independent two–electron interaction,

$$S_{\rm int} = -\frac{1}{2} \int dx_1 \, dx_2 \sum_{\sigma_1, \sigma_2} v(\mathbf{x}_1 - \mathbf{x}_2) \times \bar{\psi}_{\sigma_1}(x_1) \, \bar{\psi}_{\sigma_2}(x_2) \, \psi_{\sigma_2}(x_2) \, \psi_{\sigma_1}(x_1) \tag{2.2c}$$

and $S_{\rm dis}$ describes a static random potential $u(\mathbf{x})$ coupling to the electronic number density,

$$S_{\rm dis} = -\int dx \sum_\sigma \; u(\mathbf{x}) \, \bar{\psi}_\sigma(x) \, \psi_\sigma(x) \quad . \tag{2.2d}$$

For further calculation, we assume the random potential $u(\mathbf{x})$ in Eq. (2.2d) has variance,

$$\{u(\mathbf{x}) \, u(\mathbf{y})\}_{\rm dis} = \frac{1}{\pi N_F \tau_{\rm e}} \; \delta(\mathbf{x} - \mathbf{y}) \quad , \tag{2.3a}$$

and is Gaussian distributed,

$$\{\ldots\}_{\rm dis} = \int D[u] \; P[u] \; (\ldots) \quad , \tag{2.3b}$$

N_F is the density of states at the Fermi level, and $\tau_{\rm e}$ is the elastic scattering time. The disorder is quenched, so the replica trick[11] is used. With

$$\ln Z = \lim_{m \to 0} \; (Z^m - 1)/m \quad , \tag{2.4}$$

we consider,

$$\widetilde{Z} \equiv \{Z^m\}_{\rm dis} = \int \prod_{\alpha=1}^m D\left[\bar{\psi}^\alpha, \psi^\alpha\right] \exp[\tilde{S}\,] \quad , \tag{2.5}$$

where the corresponding action $\tilde{S}$ equals to

$$\tilde{S} = \sum_{\alpha=1}^m \left(\tilde{S}_0^\alpha + \tilde{S}_{\rm int}^\alpha + \tilde{S}_{\rm dis}^\alpha \right) \quad . \tag{2.6}$$

It is also useful to get a Fourier representation with wave vectors $\mathbf{k}$ and fermionic Matsubara frequencies $\omega_n = 2\pi T(n + 1/2)$ by the following transformations:

$$\psi_{n\sigma}(\mathbf{x}) = \sqrt{T} \int_0^\beta d\tau \; e^{i\omega_n \tau} \, \psi_\sigma(x) \quad ,$$
$$\bar{\psi}_{n\sigma}(\mathbf{x}) = \sqrt{T} \int_0^\beta d\tau \; e^{-i\omega_n \tau} \, \bar{\psi}_\sigma(x) \quad , \tag{2.7a}$$

and

$$\psi_{n\sigma}(\mathbf{k}) = \frac{1}{\sqrt{V}} \int d\mathbf{x} \; e^{-i\mathbf{k}\cdot\mathbf{x}} \, \psi_{n\sigma}(\mathbf{x}) \quad ,$$
$$\bar{\psi}_{n\sigma}(\mathbf{k}) = \frac{1}{\sqrt{V}} \int d\mathbf{x} \; e^{i\mathbf{k}\cdot\mathbf{x}} \, \bar{\psi}_{n\sigma}(\mathbf{x}) \quad . \tag{2.7b}$$

The procedure used here is similar to the one used in Ref. 6, and we refer the reader to it for further details.

B. Composite variables: Q-matrix

Now we integrate out the Grassmann fields and rewrite the theory in terms of complex-number fields. As a first step, the resulting model can then be approximately solved by using saddle-point techniques. Later fluctuations about the saddle point will be considered. First we introduce a matrix of bilinear products of the fermion fields,

$$B_{12} = \frac{i}{2} \begin{pmatrix} -\psi_{1\uparrow}\bar{\psi}_{2\uparrow} & -\psi_{1\uparrow}\bar{\psi}_{2\downarrow} & -\psi_{1\uparrow}\psi_{2\downarrow} & \psi_{1\uparrow}\psi_{2\uparrow} \\ -\psi_{1\downarrow}\bar{\psi}_{2\uparrow} & -\psi_{1\downarrow}\bar{\psi}_{2\downarrow} & -\psi_{1\downarrow}\psi_{2\downarrow} & \psi_{1\downarrow}\psi_{2\uparrow} \\ \bar{\psi}_{1\downarrow}\bar{\psi}_{2\uparrow} & \bar{\psi}_{1\downarrow}\bar{\psi}_{2\downarrow} & \bar{\psi}_{1\downarrow}\psi_{2\downarrow} & -\bar{\psi}_{1\downarrow}\psi_{2\uparrow} \\ -\bar{\psi}_{1\uparrow}\bar{\psi}_{2\uparrow} & -\bar{\psi}_{1\uparrow}\bar{\psi}_{2\downarrow} & -\bar{\psi}_{1\uparrow}\psi_{2\downarrow} & \bar{\psi}_{1\uparrow}\psi_{2\uparrow} \end{pmatrix} \cong Q_{12} \quad , \tag{2.8}$$

where all fields are understood to be taken at position $\mathbf{x}$, and $1 \equiv (n_1, \alpha_1)$ with n_1 denoting a Matsubara frequency and α a replica index, etc. The matrix elements of B commute with one another, and are therefore isomorphic to classical or complex number-valued fields that we denote by Q. We use the notation $a \cong b$ for "a is isomorphic to b". This isomorphism maps the adjoint operation on products of fermion fields, which is denoted above by an overbar, onto the complex conjugation of the classical fields. We use the isomorphism to constrain B to the classical field Q by means of a functional δ function, and exactly rewrite the partition function[6]

$$\begin{aligned} \widetilde{Z} &= \int D[\bar{\psi}, \psi] \; e^{\tilde{S}[\bar{\psi}, \psi]} \int D[Q] \, \delta[Q - B] \\ &= \int D[\bar{\psi}, \psi] \; e^{\tilde{S}[\bar{\psi}, \psi]} \int D[Q] \, D[\widetilde{\Lambda}] \; e^{{\rm Tr}\,[\widetilde{\Lambda}(Q - B)]} \\ &\equiv \int D[Q] \, D[\widetilde{\Lambda}] \; e^{\mathcal{A}[Q, \widetilde{\Lambda}]} \quad . \end{aligned} \tag{2.9}$$

Here $\widetilde{\Lambda}$ is an auxiliary bosonic matrix field that plays the role of a Lagrange multiplier, and integrates out the fermion fields.

It is useful to expand the 4×4 matrix in Eq. (2.8) in a spin-quaternion basis,

$$Q_{12}(\mathbf{x}) = \sum_{r,i=0}^{3} (\tau_r \otimes s_i)\, {}^i_r Q_{12}(\mathbf{x}) \qquad (2.10)$$

and analogously for $\widetilde{\Lambda}$. Here $\tau_0 = s_0 = \mathbb{1}_2$ is the 2×2 unit matrix, and $\tau_j = -s_j = -i\sigma_j$, $(j = 1,2,3)$, with $\sigma_{1,2,3}$ the Pauli matrices. In this basis, $i = 0$ and $i = 1,2,3$ describe the spin singlet and the spin triplet, respectively. An explicit calculation reveals that $r = 0,3$ corresponds to the particle-hole channel (i.e., products $\bar{\psi}\psi$), while $r = 1,2$ describes the particle-particle channel (i.e., products $\bar{\psi}\bar{\psi}$ or $\psi\psi$). From the structure of Eq. (2.8) one obtains the following formal symmetry properties of the Q matrices,[6]

$$ {}^0_r Q_{12} = (-)^r\, {}^0_r Q_{21} \quad , \quad (r = 0,3) \quad , \qquad (2.11a)$$
$$ {}^i_r Q_{12} = (-)^{r+1}\, {}^i_r Q_{21} \;,\; (r = 0,3;\; i = 1,2,3) \quad , \qquad (2.11b)$$
$$ {}^0_r Q_{12} = {}^0_r Q_{21} \quad , \quad (r = 1,2) \quad , \qquad (2.11c)$$
$$ {}^i_r Q_{12} = -{}^i_r Q_{21} \quad , \quad (r = 1,2;\; i = 1,2,3) \quad , \qquad (2.11d)$$
$$ {}^i_r Q^*_{12} = -{}^i_r Q^{\alpha_1\alpha_2}_{-n_1-1,-n_2-1} \quad . \qquad (2.11e)$$

Here the star in Eq. (2.11e) denotes complex conjugation.

Now by using the delta constraint in Eq. (2.9) to rewrite all terms that are quartic in the fermion field in terms of Q, we can achieve an integrand that is bilinear in ψ and $\bar{\psi}$. The Grassmannian integral can then be performed exactly, and we obtain for the effective action $\mathcal{A}$

$$\mathcal{A}[Q,\widetilde{\Lambda}] = \mathcal{A}_{\rm int}[Q] + \mathcal{A}_{\rm dis}[Q] + \frac{1}{2}\,{\rm Tr}\,\ln\left(G_0^{-1} - i\widetilde{\Lambda}\right) + \int d\mathbf{x}\;{\rm tr}\,\left(\widetilde{\Lambda}(\mathbf{x})\, Q(\mathbf{x})\right) \quad . \qquad (2.12)$$

Here Tr denotes a trace over all degrees of freedom, including the continuous position variable, while tr is a trace over all those discrete indices that are not explicitly shown. And

$$G_0^{-1} = -\partial_\tau + \partial^2_{\mathbf{x}}/2m + \mu \qquad (2.13)$$

is the inverse free electron Green operator, with ∂_τ and $\partial_{\mathbf{x}}$ derivatives with respect to imaginary time and position, respectively, m is the electron mass, and μ is the chemical potential. We can see from the structure of the Tr ln-term in Eq. (2.12) that the physical meaning of the auxiliary field $\widetilde{\Lambda}$ is that of a self–energy. The electron-electron interaction $\mathcal{A}_{\rm int}$ is conveniently decomposed into four pieces that describe the interaction in the particle-hole and particle-particle spin-singlet and spin-triplet channels.[6] For the purposes of the present paper, we need only the particle-particle spin-singlet channel interaction explicitly to describe superconductivity. Similar to the BCS model we ignore the normal Coulomb repulsion in the particle-hole channels, and we also ignore the possibility of triplet superconductivity.[12] Then

$$\begin{aligned}\mathcal{A}_{\rm int}[Q] &= \mathcal{A}^{(c)}_{\rm int} \\ &= \frac{T\Gamma^{(c)}}{2}\int d\mathbf{x} \sum_{r=1,2}\sum_{n_1,n_2,m}\sum_{\alpha} \\ &\quad \times \left[{\rm tr}\,\left((\tau_r\otimes s_0)\, Q^{\alpha\alpha}_{n_1,-n_1+m}(\mathbf{x})\right)\right] \\ &\quad \times \left[{\rm tr}\,\left((\tau_r\otimes s_0)\, Q^{\alpha\alpha}_{-n_2,n_2+m}(\mathbf{x})\right)\right] \quad , \end{aligned} \qquad (2.14)$$

with $\Gamma^{(c)}$ the particle-particle spin-singlet channel interaction amplitude, with $\Gamma^{(c)} < 0$ leading to superconductivity. For the disorder part of the effective action one finds[13]

$$\mathcal{A}_{\rm dis}[Q] = \frac{1}{\pi N_F \tau_{\rm e}}\int d\mathbf{x}\;{\rm tr}\,\left(Q(\mathbf{x})\right)^2 \quad . \qquad (2.15)$$

We will focus on the matrix elements ${}^0_0Q$ and ${}^0_1Q$ in disordered superconductivity states. From Eqs. (2.8) and (2.10) we find

$$\begin{aligned}{}^0_0Q_{12}(\mathbf{x}) \cong \frac{i}{8}\; &\left[-\psi_{1\uparrow}(\mathbf{x})\bar{\psi}_{2\uparrow}(\mathbf{x}) - \psi_{1\downarrow}(\mathbf{x})\bar{\psi}_{2\downarrow}(\mathbf{x}) \right.\\ &\left. + \bar{\psi}_{1\downarrow}(\mathbf{x})\psi_{2\downarrow}(\mathbf{x}) + \bar{\psi}_{1\uparrow}(\mathbf{x})\psi_{2\uparrow}(\mathbf{x})\right] \quad , \end{aligned} \qquad (2.16a)$$

$$\begin{aligned}{}^0_1Q_{12}(\mathbf{x}) \cong \frac{-1}{8}\; &\left[-\psi_{1\uparrow}(\mathbf{x})\psi_{2\downarrow}(\mathbf{x}) + \psi_{1\downarrow}(\mathbf{x})\psi_{2\uparrow}(\mathbf{x}) \right.\\ &\left. + \bar{\psi}_{1\downarrow}(\mathbf{x})\bar{\psi}_{2\uparrow}(\mathbf{x}) - \bar{\psi}_{1\uparrow}(\mathbf{x})\bar{\psi}_{2\downarrow}(\mathbf{x})\right] \quad . \end{aligned} \qquad (2.16b)$$

Note that ${}^0_2Q_{12}$ has a structure similar to ${}^0_1Q_{12}$. This implies we could use ${}^0_2Q_{12}$ instead of ${}^0_1Q_{12}$. Physically, ${}^0_0Q_{12}$ is related to the single particle density of states, while ${}^0_1Q_{12}$ is basically the superconducting order parameter.

III. SADDLE–POINT SOLUTIONS AND GAUSSIAN APPROXIMATION

A. The saddle–point method

We now look for a saddle-point solution of the field theory derived in the previous section. The saddle–point condition is[6,14]

$$\left.\frac{\delta\mathcal{A}}{\delta Q}\right|_{Q_{\rm sp},\widetilde{\Lambda}_{\rm sp}} = \left.\frac{\delta\mathcal{A}}{\delta\widetilde{\Lambda}}\right|_{Q_{\rm sp},\widetilde{\Lambda}_{\rm sp}} = 0 \quad . \qquad (3.1)$$

According to Eqs. (2.16), the saddle point values of both Q and $\widetilde{\Lambda}$ in singlet superconductivity–like phases have the structures

$$\begin{aligned}\left.{}^i_rQ_{12}(\mathbf{x})\right|_{\rm sp} &= \delta_{\alpha_1\alpha_2}\,\delta_{i0}\;\left[\delta_{n_1,-n_2}\,\delta_{r1}\,Q_{n_1} \right.\\ &\quad\left. + \delta_{n_1,n_2}\,\delta_{r0}\,\Lambda_{n_1}\right] \quad , \end{aligned} \qquad (3.2a)$$

$$\begin{aligned}\left.{}^i_r\widetilde{\Lambda}_{12}(\mathbf{x})\right|_{\rm sp} &= \delta_{\alpha_1\alpha_2}\,\delta_{i0}\;\left[\delta_{n_1,-n_2}\,\delta_{r1}\,(iq_{n_1}) \right.\\ &\quad\left. + \delta_{n_1,n_2}\,\delta_{r0}\,(-i\lambda_{n_1})\right] \quad . \end{aligned} \qquad (3.2b)$$

where we assume $\Lambda_n = -\Lambda_{-n}$, $\lambda_n = -\lambda_{-n}$ which is equivalent to a redefinition of the chemical potential,[14]

and set $Q_n = Q_{-n}$, $q_n = q_{-n}$ which follows from Eqs. (3.2) and (2.11c). Substituting this into Eqs. (2.12) - (2.15), and using the saddle-point condition Eq. (3.1), we obtain the saddle-point equations

$$\Lambda_n = \frac{i}{2V}\sum_{\mathbf{k}} \mathcal{G}_n(\mathbf{k}) \quad , \tag{3.3a}$$

$$Q_n = \frac{-i}{2V}\sum_{\mathbf{k}} \mathcal{F}_n(\mathbf{k}) \quad , \tag{3.3b}$$

$$\lambda_n = \frac{-2i}{\pi N_{\rm F}\tau_{\rm e}}\,\Lambda_n \quad , \tag{3.3c}$$

$$q_n = \frac{2i}{\pi N_{\rm F}\tau_{\rm e}}\,Q_n - 4i\,\Gamma^{(c)}\,T\sum_m Q_m \quad . \tag{3.3d}$$

Here

$$\mathcal{G}_n(\mathbf{k}) = \frac{-(i\omega_n - \lambda_n) - \xi_{\mathbf{k}}}{-(i\omega_n - \lambda_n)^2 + \xi_{\mathbf{k}}^2 + q_n^2} \quad , \tag{3.4a}$$

$$\mathcal{F}_n(\mathbf{k}) = \frac{q_n}{-(i\omega_n - \lambda_n)^2 + \xi_{\mathbf{k}}^2 + q_n^2} \quad , \tag{3.4b}$$

are Green functions with $\xi_{\mathbf{k}} = \mathbf{k}^2/2m - \mu$.

From Eqs. (3.3), it is easy to find

$$\lambda_n = \frac{1}{\pi N_{\rm F}\tau_{\rm e}}\,\frac{1}{V}\sum_{\mathbf{k}} \mathcal{G}_n(\mathbf{k}) \quad , \tag{3.5a}$$

$$\begin{aligned} q_n = {} & \frac{1}{\pi N_{\rm F}\tau_{\rm e}}\,\frac{1}{V}\sum_{\mathbf{k}} \mathcal{F}_n(\mathbf{k}) \\ & -2\,\Gamma^{(c)}\,T\,\frac{1}{V}\sum_{\mathbf{k}}\sum_m \mathcal{F}_m(\mathbf{k}) \quad . \end{aligned} \tag{3.5b}$$

We now define a gap function Δ by[15]

$$q_n = \bar{q}_n + \Delta \equiv \eta_n\Delta \quad , \tag{3.6a}$$

with

$$\bar{q}_n = \frac{1}{\pi N_{\rm F}\tau_{\rm e}}\,\frac{1}{V}\sum_{\mathbf{k}} \mathcal{F}_n(\mathbf{k}) \quad , \tag{3.6b}$$

and it can be shown that

$$\eta_n\omega_n = i\lambda_n + \omega_n \quad . \tag{3.6c}$$

We then obtain the gap equation,

$$\begin{aligned} \Delta &= -2\,\Gamma^{(c)}\,T\,\frac{1}{V}\sum_{\mathbf{k}}\sum_n \frac{\eta_n\Delta}{(\eta_n\omega_n)^2 + \xi_{\mathbf{k}}^2 + (\eta_n\Delta)^2} \\ &= -2\,\Gamma^{(c)}\,T\sum_n N(0)\int d\xi_{\mathbf{k}}\,\frac{\Delta}{\omega_n^2 + \xi_{\mathbf{k}}^2 + \Delta^2} \end{aligned} \tag{3.7}$$

with $N(0) = \frac{N_F}{2}$ the density of states per spin at the Fermi surface. A remarkable aspect of this gap equation is that in this approximation the gap Δ and the critical temperature T_c are independent of the (nonmagnetic) disorder, and so are all thermodynamic properties in superconductivity. This result is known as Anderson's theorem.[16]

We next obtain the density of states. From Eq. (2.16a) it follows,

$$N(\epsilon_F + \omega) = \frac{4}{\pi}\,\mathrm{Re}\left\langle {}^0_0Q_{nn}(\mathbf{x})\right\rangle\Big|_{i\omega_n\to\omega+i0} \quad . \tag{3.8}$$

In saddle point approximation, we have for the density of states

$$\begin{aligned} N(\epsilon_F + \omega) &= \frac{-2}{\pi}\,\frac{1}{V}\sum_{\mathbf{k}} \mathrm{Im}\,\mathcal{G}_n(\mathbf{k}, i\omega_n \to \omega + i0) \\ &= N_F\,\frac{\omega}{\sqrt{\omega^2 - \Delta^2}} \quad \text{for } \omega > \Delta \\ &= 0 \quad \text{for } \omega < \Delta \quad . \end{aligned} \tag{3.9}$$

For later reference we also define a matrix saddle-point Green function

$$G_{\rm sp} = \left(G_0^{-1} - i\widetilde{\Lambda}\right)^{-1}\Big|_{\rm sp} \quad , \tag{3.10a}$$

whose matrix elements are given by

$$\begin{aligned} (G_{\rm sp})_{nm}(\mathbf{k}) = {} & \delta_{nm}\,\mathcal{G}_n(\mathbf{k})\,(\tau_0\otimes s_0) \\ & - \delta_{n,-m}\,\mathcal{F}_n(\mathbf{k})\,(\tau_1\otimes s_0) \quad . \end{aligned} \tag{3.10b}$$

Note that the above results are the standard ones.

B. Gaussian approximation

We next set up the calculation of the Gaussian fluctuations about the saddle point discussed above. In the following section these results will be used to compute the physical correlation functions in the disordered superconducting phase. To this end, we write Q and $\widetilde{\Lambda}$ in Eqs. (2.12) - (2.15) as,

$$Q = Q_{\rm sp} + \delta Q \quad , \tag{3.11a}$$

$$\widetilde{\Lambda} = \widetilde{\Lambda}_{\rm sp} + \delta\widetilde{\Lambda} \quad , \tag{3.11b}$$

and then expand to second or Gaussian order in the fluctuations δQ and $\delta\widetilde{\Lambda}$. Denoting the constant saddle point contribution to the effective action by $\mathcal{A}_{\rm sp}$, and the Gaussian action by $\mathcal{A}_G$, we have, to the Gaussian order, that

$$\mathcal{A}[Q,\widetilde{\Lambda}] = \mathcal{A}_{\rm sp} + \mathcal{A}_G[Q,\widetilde{\Lambda}] \quad , \tag{3.12}$$

with

$$\begin{aligned} \mathcal{A}_G[Q,\widetilde{\Lambda}] = {} & \mathcal{A}_{\rm int}[\delta Q] + \mathcal{A}_{\rm dis}[\delta Q] + \frac{1}{4}\,\mathrm{Tr}\,\left(G_{\rm sp}\delta\widetilde{\Lambda}G_{\rm sp}\,\delta\widetilde{\Lambda}\right) \\ & + \int d\mathbf{x}\;\mathrm{tr}\,\left(\delta\widetilde{\Lambda}(\mathbf{x})\,\delta Q(\mathbf{x})\right) \quad , \end{aligned} \tag{3.13}$$

For the quadratic part we find

$$\frac{1}{4}\mathrm{Tr}\,\left(G_{\mathrm{sp}}\,\delta\widetilde{\Lambda}\,G_{\mathrm{sp}}\,\delta\widetilde{\Lambda}\right) = \frac{1}{V}\sum_{\mathbf{k}}\sum_{1,2,3,4}\sum_{r,s}\sum_{i,j}{}^{i}_{r}(\delta\widetilde{\Lambda})_{12}(\mathbf{k}) \times {}^{ij}_{rs}A_{12,34}(\mathbf{k})\,{}^{j}_{s}(\delta\widetilde{\Lambda})_{34}(-\mathbf{k}) \quad . \tag{3.14a}$$

Here

$$\begin{aligned}{}^{ij}_{rs}A_{12,34}(\mathbf{k}) &= \delta_{13}\,\delta_{24}\,\varphi^{00}_{12}(\mathbf{k})\,N^{00}_{rs}\,\delta_{ij}\,{}^{i}_{r}I_{12} \\ &+\delta_{13}\,\delta_{2,-4}\,\varphi^{01}_{12}(\mathbf{k})\,N^{01}_{rs}\,\delta_{ij}\,{}^{i}_{r}I_{12} \\ &+\delta_{1,-3}\,\delta_{24}\,\varphi^{10}_{12}(\mathbf{k})\,N^{10}_{rs}\,\delta_{ij}\,{}^{i}_{r}I_{12} \\ &+\delta_{1,-3}\,\delta_{2,-4}\,\varphi^{11}_{12}(\mathbf{k})\,N^{11}_{rs}\,\delta_{ij}\,{}^{i}_{r}I_{12}, \\ &\equiv {}^{ij}_{rs}A^{(0)}_{12,34}(\mathbf{k})\,{}^{i}_{r}I_{12} \quad ,\end{aligned} \tag{3.14b}$$

with 4×4 matrices

$$N^{00} = \begin{pmatrix} i\tau_3 & 0 \\ 0 & -i\tau_3 \end{pmatrix} \quad , \quad N^{01} = \begin{pmatrix} -i\tau_1 & 0 \\ 0 & -i\tau_1 \end{pmatrix} \quad ,$$
$$N^{10} = \begin{pmatrix} -i\tau_1 & 0 \\ 0 & i\tau_1 \end{pmatrix} \quad , \quad N^{11} = \begin{pmatrix} -i\tau_3 & 0 \\ 0 & -i\tau_3 \end{pmatrix} \quad , \tag{3.14c}$$

and

$${}^{i}_{r}I_{12} = 1 + \delta_{12}\left[-1 + \begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_i\right] \quad , \tag{3.14d}$$

where $\begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}_r = \delta_{r0} + \delta_{r1} + \delta_{r2} - \delta_{r3}$, etc. and

$$\varphi^{00}_{nm}(\mathbf{k}) = \frac{1}{V}\sum_{\mathbf{p}}\mathcal{G}_n(\mathbf{p})\,\mathcal{G}_m(\mathbf{p}+\mathbf{k}) \quad , \tag{3.14e}$$

and φ^{01}, φ^{10}, and φ^{11} defined similarly with $\mathcal{G}\mathcal{G}$ in Eq. (3.14e) replaced by $(-1)\mathcal{G}\mathcal{F}$, $(-1)\mathcal{F}\mathcal{G}$, and $\mathcal{F}\mathcal{F}$, respectively.

In a similar way, the term that couples $\delta\widetilde{\Lambda}$ and δQ can be written

$$\mathrm{Tr}\,\left(\delta\widetilde{\Lambda}\,\delta Q\right) = 4\sum_{1,2,3,4}\frac{1}{V}\sum_{\mathbf{k}}\sum_{r,i}{}^{i}_{r}(\delta\widetilde{\Lambda})_{12}(\mathbf{k}) \times {}^{i}_{r}B_{12}(\mathbf{k})\,{}^{i}_{r}(\delta Q)_{12}(-\mathbf{k})\ , \tag{3.15a}$$

where

$${}^{i}_{r}B_{12}(\mathbf{k}) = {}^{i}_{r}I_{12}\begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_r \quad . \tag{3.15b}$$

Q and $\widetilde{\Lambda}$ can now be decoupled by shifting and scaling the $\widetilde{\Lambda}$ field. If we define a new field $\bar{\Lambda}$ by

$${}^{i}_{r}(\delta\widetilde{\Lambda})_{12}(\mathbf{k}) = 2\,{}^{ij}_{rs}(A^{-1})_{12,34}(\mathbf{k}) \times \left({}^{j}_{s}(\delta\bar{\Lambda})_{34}(\mathbf{k}) - {}^{j}_{s}(\delta Q)_{34}(\mathbf{k})\right){}^{j}_{s}B_{34}, \tag{3.16}$$

with A^{-1} being the inverse of the matrix A defined in Eq. (3.14b), then $\bar{\Lambda}$ and Q decouple. Integrating out the $\delta\bar{\Lambda}$ fluctuations leads to a Gaussian action completely in terms of δQ fluctuations,

$$\begin{aligned}\mathcal{A}_{\mathrm{G}}[Q] = -\frac{4}{V}\sum_{\mathbf{k}}\sum_{1234}\sum_{rs}\sum_{ij}{}^{i}_{r}(\delta Q)_{12}(\mathbf{k})\,{}^{ij}_{rs}(A^{-1})_{12,34}(\mathbf{k}) \\ \times\,{}^{i}_{r}B_{12}\,{}^{j}_{s}B_{34}\,{}^{j}_{s}(\delta Q)_{34}(-\mathbf{k}) \\ +\mathcal{A}_{\mathrm{int}}[\delta Q] + \mathcal{A}_{\mathrm{dis}}[\delta Q] \quad ,\end{aligned} \tag{3.17}$$

It is convenient to rewrite this result as

$$\mathcal{A}_{\mathrm{G}}[Q] = \frac{-4}{V}\sum_{\mathbf{k}}\sum_{1234}\sum_{rs}\sum_{ij}{}^{i}_{r}(\delta Q)_{12}(\mathbf{k})\,{}^{ij}_{rs}M_{12,34}(\mathbf{k}) \times {}^{j}_{s}(\delta Q)_{34}(-\mathbf{k}) \quad , \tag{3.18a}$$

where

$$\begin{aligned}{}^{ij}_{rs}M_{12,34}(\mathbf{k}) &= {}^{ij}_{rs}(A^{-1})_{12,34}(\mathbf{k})\,{}^{i}_{r}B_{12}\,{}^{j}_{s}B_{34} \\ &-2T\Gamma^{(c)}\,\delta_{ij}\delta_{rs}\delta_{1+2,3+4}\begin{pmatrix} 0 \\ + \\ + \\ 0 \end{pmatrix}_r \begin{pmatrix} + \\ 0 \\ 0 \\ 0 \end{pmatrix}_i \\ &-\frac{1}{\pi N_F\tau_{\mathrm{e}}}{}^{i}_{r}B_{12}\,\delta_{ij}\delta_{rs}\delta_{13}\delta_{24}\ .\end{aligned} \tag{3.18b}$$

IV. PHYSICAL CORRELATION FUNCTIONS

A. Ultrasonic attenuation by saddle-point approximation

We now use the results of the preceding sections to calculate transverse ultrasonic attenuation in both clean and disordered superconductors. As shown in Ref. 17, the sound attenuation coefficient has the expression

$$\alpha(\omega) = \lim_{k\to 0}\frac{\omega}{\rho_{ion}c_s^3}\,\mathrm{Im}\chi(\mathbf{k}, i\omega_n \to \omega + i0) \quad , \tag{4.1}$$

where, with $D_x \equiv \partial_{x_1}\partial_{x_2}$,

$$\begin{aligned}\chi(\mathbf{k}, i\omega_n) = \frac{1}{m_e^2}\frac{1}{V}\int d\mathbf{x}d\mathbf{x}'d\mathbf{y}d\mathbf{y}'\exp\left(-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\right)\sum_{\sigma_1,\sigma_2}\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{y}-\mathbf{y}')D_xD_y \\ \times\frac{1}{\beta}\sum_{\omega_1,\omega_2}\langle\bar{\psi}^{\alpha}_{\omega_1,\sigma_1}(\mathbf{x})\psi^{\alpha}_{\omega_1-\omega_n,\sigma_1}(\mathbf{x}')\bar{\psi}^{\alpha}_{\omega_2,\sigma_2}(\mathbf{y})\psi^{\alpha}_{\omega_2+\omega_n,\sigma_2}(\mathbf{y}')\rangle \quad .\end{aligned} \tag{4.2}$$

By introducing a source term of the form

$$\delta\tilde{S}^{\alpha} = \int d\mathbf{x} \sum_{\omega_n} h(\omega_n, \mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \sum_{\omega,\sigma} \bar{\psi}^{\alpha}_{\omega,\sigma}(\mathbf{x})\, D_x\, \psi^{\alpha}_{\omega+\omega_n,\sigma}(\mathbf{x}), \tag{4.3}$$

we can obtain

$$\chi(\mathbf{k}=0, i\omega_n) = \frac{1}{m_e^2 \beta V} \frac{\partial^2 \widetilde{Z}}{\partial h(\omega_n, \mathbf{k}) \partial h(\omega_{-n}, -\mathbf{k})}\bigg|_{h=0} \tag{4.4}$$

with the third term of the right side of the Eq. (2.12) becoming

$$\begin{aligned} \mathcal{A}_3 &= \frac{1}{2} \mathrm{Tr}\ \ln\left(G_0^{-1} - i\widetilde{\Lambda} + D\right) \\ &= \frac{1}{2} \mathrm{Tr}\ \ln\left(G_0^{-1} - i(\widetilde{\Lambda}_{sp} + \delta\widetilde{\Lambda}) + D\right) \\ &= \frac{1}{2} \mathrm{Tr}\ \ln\left(1 + DG_{sp} - i\delta\widetilde{\Lambda} G_{sp}\right) + \frac{1}{2} \mathrm{Tr}\ \ln\left(G_{sp}^{-1}\right) \end{aligned} \tag{4.5}$$

and $D \equiv \sum_{\omega_n} \delta(\omega_1 - \omega_2 + \omega_n) h \exp\left(-i\mathbf{k}\cdot\mathbf{x}\right) D_x$.

In the saddle-point approximation, we neglect the $\delta\widetilde{\Lambda}$ item and have

$$\mathcal{A}_3 = \frac{-1}{4} \mathrm{Tr}\ (DG_{sp}DG_{sp}) + const. \tag{4.6}$$

We then obtain the ultrasonic attenuation coefficient, for small frequency,

$$\alpha_s(\omega) = \alpha_n \frac{2}{1 + \exp\left(\beta\Delta\right)} \tag{4.7}$$

for both clean and disordered superconductors. Here α_n is the attenuation coefficient of the normal metal.[18] In the clean metal it has

$$\alpha_{n,clean} = \frac{k_f^4 \omega^2}{30\pi q \rho_{ion} c_s^3} \tag{4.8a}$$

with the usual conditions $\omega < qv_f < \Delta$ satisfied. In the disordered case

$$\alpha_{n,disordered} = \frac{2N(0) k_f^4 \omega^2 \tau}{15 m^2 \rho_{ion} c_s^3}, \tag{4.8b}$$

where the approximation of $\tau_{\mathrm{e}}\Delta \ll 1$ is assumed, which is called the dirty limit.[19] The above result confirms the one of Levy's which was obtained by Boltzmann's transport equation.[20] It is noted that no Green function method has been used to obtain this result before.

The above method can be used to obtain other physical properties, like longitudinal electrical conductivity. In that case, higher-order corrections must be included due to the gauge invariance problem.[21] Below we show how to correctly obtain the conductivity by using the Gaussian fluctuations about the saddle point.

B. Physical correlation functions by Gaussian fluctuations

1. Gaussian propagators

We now expand our method to calculate the Gaussian propagators and then to obtain the number density susceptibility, χ_n, the spin density susceptibility, χ_s and the conductivity. We find in Appendix A that the number density susceptibility, χ_n, and the spin density susceptibility, χ_s, can be expressed in terms of the Q-correlation functions,

$$\begin{aligned} \chi^{(i)}(\mathbf{k}, \omega_n) = \frac{16T}{V} \sum_{1,2} \sum_{r=0,3} \Big\langle {}^{i}_{r}(\delta Q)_{1+n,1}(\mathbf{k}) \\ \times {}^{i}_{r}(\delta Q)_{2+n,2}(-\mathbf{k}) \Big\rangle \quad , \end{aligned} \tag{4.9}$$

with $\chi^{(0)} = \chi_n$ and $\chi^{(1,2,3)} = \chi_s$. Here the Gaussian propagators in the Eq. (4.9) are given in terms of the inverse of the matrix M defined in Eq. (3.18b) by

$$\begin{aligned} \Big\langle {}^{i}_{r}(\delta Q)_{12}(\mathbf{k}_1)\, {}^{j}_{s}(\delta Q)_{34}(\mathbf{k}_2) \Big\rangle_G = \frac{V}{8}\, \delta_{\mathbf{k}_1, -\mathbf{k}_2} \\ \times {}^{ij}_{rs}M^{-1}_{12,34}(\mathbf{k}_1)\ , \end{aligned} \tag{4.10}$$

where $\langle \ldots \rangle_G$ denotes an average with the Gaussian action $\mathcal{A}_G$. We see from Eqs. (4.9) and (4.10) that M^{-1} determines the correlation functions within Gaussian approximation.

In the following section we will be interested in the number density susceptibility χ_n. Other correlation functions can be obtained similarly by applying the technique introduced below. From the expression of Q in terms of the fermion fields, Eq. (2.8), it is easy to see that the contributions to Eq. (4.9) from $r=0$ and $r=3$ are identical for $\omega_n \neq 0$. We can therefore write

$$\chi_n(\mathbf{k}, \omega_n) = 4T \sum_{1,2} {}^{00}_{33}M^{-1}_{1+n,1;2+n,2}(\mathbf{k}) \quad , \tag{4.11}$$

To find $\sum_{1,2} {}^{00}_{33}M^{-1}_{1+n,1;2+n,2}$, we rewrite M as

$$\begin{aligned} {}^{ij}_{rs}M_{12,34}(\mathbf{k}) &\equiv {}^{ij}_{rs}(A^{-1})_{12,34}(\mathbf{k})\, {}^{i}_{r}B_{12}\, {}^{j}_{s}B_{34} \\ &\quad - {}^{ij}_{rs}D_{12,34} \\ &\equiv {}^{ij}_{rs}(C^{-1})_{12,34}(\mathbf{k}) \\ &\quad - {}^{ij}_{rs}D_{12,34} \quad . \end{aligned} \tag{4.12}$$

Then we find

$$M^{-1} = \left(C^{-1} - D\right)^{-1} \quad . \tag{4.13}$$

It is convenient to write the inverse of the matrix M as an integral equation,

$$M^{-1} = C + C\,D\,M^{-1} \quad , \tag{4.14}$$

with

$${}^{ij}_{rs}C_{12,34} = {}^{ij}_{rs}A^{(0)}_{12,34} \begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_r {}^{j}_{s}B_{34} \quad . \tag{4.15}$$

For further simplicity, we set $\Gamma = 2T\Gamma^{(c)}$, $\tau^0 = \pi N_F \tau_{\mathrm{e}}$ and ${}^{i}_{r}I_{12} = 1$ for $\omega_n \neq 0$. Expanding Eq. (4.14) we have

$${}^{00}_{33}M^{-1}_{12,34} = {}^{00}_{33}A^{(0)}_{12,34} - \Gamma \begin{pmatrix} -\varphi^{01}_{12} \sum_{78} \delta_{1-2,7+8}\, {}^{00}_{23}M^{-1}_{78,34} \\ +\varphi^{10}_{12} \sum_{78} \delta_{-1+2,7+8}\, {}^{00}_{23}M^{-1}_{78,34} \end{pmatrix} + \frac{1}{\tau^0} \begin{pmatrix} +\varphi^{00}_{12}\, {}^{00}_{33}M^{-1}_{1,2;3,4} \\ -\varphi^{01}_{12}\, {}^{00}_{23}M^{-1}_{1,-2;3,4} \\ +\varphi^{10}_{12}\, {}^{00}_{23}M^{-1}_{-1,2;3,4} \\ +\varphi^{11}_{12}\, {}^{00}_{33}M^{-1}_{-1,-2;3,4} \end{pmatrix} \quad , \tag{4.16}$$

where we have used the structures of B, Eq. (3.15b) and $A^{(0)}$, Eq. (3.14b). ${}^{10}_{23}M^{-1}$ in turn obeys the integral equation

$${}^{00}_{23}M^{-1}_{12,34} = -{}^{00}_{23}A^{(0)}_{12,34} + \Gamma \begin{pmatrix} -\varphi^{00}_{12} \sum_{78} \delta_{1+2,7+8}\, {}^{00}_{23}M^{-1}_{78,34} \\ -\varphi^{11}_{12} \sum_{78} \delta_{-1-2,7+8}\, {}^{00}_{23}M^{-1}_{78,34} \end{pmatrix} - \frac{1}{\tau^0} \begin{pmatrix} -\varphi^{00}_{12}\, {}^{00}_{23}M^{-1}_{1,2;3,4} \\ -\varphi^{01}_{12}\, {}^{00}_{33}M^{-1}_{1,-2;3,4} \\ +\varphi^{10}_{12}\, {}^{00}_{33}M^{-1}_{-1,2;3,4} \\ -\varphi^{11}_{12}\, {}^{00}_{23}M^{-1}_{-1,-2;3,4} \end{pmatrix} \quad . \tag{4.17}$$

Similar results can be obtained for ${}^{00}_{33}M^{-1}_{-1,-2;3,4}$ and ${}^{00}_{23}M^{-1}_{-1,-2;3,4}$.

We can now obtain the $\sum_{1,2} {}^{00}_{33}M^{-1}_{1+n,1;2+n,2}$ now. Obviously $\sum_{1,2} {}^{00}_{23}M^{-1}_{1+n,-1;2+n,2}$ and $\sum_{1,2} {}^{00}_{23}M^{-1}_{-1-n,1;2+n,2}$ need to be determined first. We find in Eq. (4.17) that all of the ${}^{00}_{23}M^{-1}$ form a linear equation group which can be solved by using Cramer's Rule. It is then easy to obtain ${}^{00}_{33}M^{-1}_{1+n,1;2+n,2}$ by Eq. (4.16). Through Eq. (4.11) the number density susceptibility χ_n can finally be evaluated explicitly. Note that this technique can then be generalized to obtain all elements of M^{-1}, which in turn gives the Gaussian propagators or physical correlation functions completely.

2. Correlation functions in the clean limit

In this section we discuss the clean limit, or the non-impurity electron gas. Let us perform the clean limit, $\tau_{\mathrm{e}} \to \infty$. $\mathcal{A}_{\mathrm{dis}}$ then vanishes. That also means $\lambda_n \to 0$, $\Lambda_n \to 0$ and $q_n = \Delta$.

For small $|\mathbf{k}|$ and ω_n, we obtain the number density susceptibility of clean superconductor,

$$\chi_n(\mathbf{k}, \omega_n) = -N_F\, \frac{\frac{v_f^2}{3}\mathbf{k}^2}{\omega_n^2 + \frac{v_f^2}{3}\mathbf{k}^2} \quad . \tag{4.18}$$

The electrical conductivity σ is determined by χ_n via[22]

$$\sigma(\mathbf{k}, \omega) = i e^2 \frac{\omega}{\mathbf{k}^2}\, \chi_n(\mathbf{k}, i\omega_n \to \omega + i0) \quad . \tag{4.19}$$

In particular, the real part of the conductivity as a function of real frequencies has a delta-function contribution

$$\begin{aligned} \mathrm{Re}\ \sigma(\omega) &= -\lim_{k \to 0} e^2 \frac{\omega}{\mathbf{k}^2}\, \mathrm{Im}\chi_n(\mathbf{k}, i\omega_n \to \omega + i0) \\ &= \frac{e^2\, N_F\, \pi\, v_f^2}{3}\, \delta(\omega) \\ &= \frac{n\, \pi\, e^2}{m}\, \delta(\omega) \quad , \end{aligned} \tag{4.20}$$

with $n = \frac{k_f^3}{3\pi^2}$ the particle number density. This coincides with the result already known.[23–25]

Similar procedure can be applied to obtain the spin density susceptibility, by noting that

$$\begin{aligned} \chi_s(\mathbf{k}, \omega_n = 0) &= \frac{16T}{V} \sum_{1,2} \left\langle \tfrac{1}{3}(\delta Q)_{1,1}(\mathbf{k})\, \tfrac{1}{3}(\delta Q)_{2,2}(-\mathbf{k}) \right\rangle \\ &= 2T \sum_{1,2} {}^{11}_{33}M^{-1}_{1,1;2,2} \end{aligned} \tag{4.21a}$$

and

$$\begin{aligned} \sum_{1,2} {}^{11}_{33}M^{-1}_{1+n,1;2+n,2} &= \sum_{1,2} {}^{11}_{33}A^{(0)}_{1+n,1;2+n,2} \\ &= \frac{-N_F}{2T}\, \frac{n_n}{n} \\ &\quad \text{for } \omega_n = 0,\ |\mathbf{k}| \to 0, \end{aligned} \tag{4.21b}$$

where $n = n_s + n_n$, with n_s the density of superconducting electrons, n_n the density of normal electrons,[26]

$$n_n = n \int_{-\infty}^{\infty} d\xi_{\mathbf{p}} \frac{\exp\left(\frac{\sqrt{\xi_{\mathbf{p}}{}^2+\Delta^2}}{T}\right)}{T\left(1+\exp\left(\frac{\sqrt{\xi_{\mathbf{p}}{}^2+\Delta^2}}{T}\right)\right)^2}. \quad (4.21c)$$

The result $\chi_s(k \to 0, \omega_n = 0) = -N_F \frac{n_n}{n}$ is consistent with Yosida's.[27]

The above result means $\chi_s = 0$ at zero temperature. This is because a BCS superconductor is a perfect diamagnet at $T = 0$.[27] The non-zero part comes from the contribution of normal electrons at finite temperature,[28] since some Cooper pairs are broken into normal electrons at $T \neq 0$.

3. Correlation functions in the disordered case

Now we turn to the disordered case. The approximation of $\tau_e\Delta \ll 1$ is again assumed. Calculations at $T \to 0$ show that in the limit of long wavelength and low frequency,

$$\chi_n(\mathbf{k}, \omega_n) = -N_F \frac{\frac{\pi\Delta\tau_e v_f^2}{3}\mathbf{k}^2}{\omega_n^2 + \frac{\pi\Delta\tau_e v_f^2}{3}\mathbf{k}^2} \quad , \quad (4.22)$$

and the real part of the conductivity as a function of real frequencies has also a delta-function contribution

$$\begin{aligned} \mathrm{Re}\ \sigma(\omega \to 0) &= -\lim_{k\to 0} e^2 \frac{\omega}{\mathbf{k}^2} \mathrm{Im}\chi_n(\mathbf{k}, i\omega_n \to \omega + i0) \\ &= \frac{e^2\, N_F\, \Delta\, \tau_e\, \pi^2\, v_f^2}{3}\, \delta(\omega) \quad . \quad (4.23) \end{aligned}$$

Note that to satisfy the f-sum rule in the disordered case the conductivity will not vanish completely at finite frequency. Our calculation shows that at $T = 0$,

$$\begin{aligned} \mathrm{Re}\ \sigma(\omega > 2\Delta) &= \frac{\sigma_n}{\omega} \\ &\times \int_{\Delta}^{\omega-\Delta} dE \frac{-E(E-\omega) - \Delta^2}{\sqrt{E^2-\Delta^2}\sqrt{(E-\omega)^2-\Delta^2}}, \quad (4.24) \end{aligned}$$

where the vertex corrections resulting from the impurity scattering and the interaction have been omitted. σ_n is the conductivity of normal metal. This coincides with the result already known, too.[25]

Again, similar procedure can be applied to obtain the spin density susceptibility. We find that, at $T = 0$

$$\chi_s(k \to 0, \omega_n = 0) = 0 \quad , \quad (4.25)$$

That means the spin response in the nonmagnetic disordered case is the same as that in the clean limit. This is consistent with Devereaux and Belitz's argument,[29] which has shown that the nonmagnetic disorder has no effect on the spin–flip pair breaking rate.

V. CONCLUSION

In this paper we presented a method to study the transport properties of disordered s–wave superconductor. The crucial idea is to first identify the saddle points of the system by using a symmetry analysis, then to study the fluctuations around them to obtain the physical correlation functions. The ultrasonic attenuation, number density susceptibility, the spin density susceptibility and the conductivity have been calculated in the clean superconductor, as well as in the disordered superconductor. Other properties, like energy correlation function, can be similarly obtained. Furthermore, the formalism here can be a powerful tool to study the quantum phase transitions between normal metal and superconductor.

Finally, we remark, that the techniques used here can be used to study gapless s–wave superconductors, as well as, for example, disordered d–wave SC relevant to the high T_c superconductors.[30]

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APPENDIX A: CORRELATION FUNCTIONS IN TERMS OF Q MATRICES

The real number density susceptibility has the following form[2]

$$X^R(\mathbf{x}_1 t_1, \mathbf{x}_2 t_2) = -i\theta(t_1 - t_2)\langle[\tilde{n}(\mathbf{x}_1 t_1), \tilde{n}(\mathbf{x}_2 t_2)]\rangle \quad (A1)$$

where

$$\tilde{n} = n - \langle n \rangle \quad (A2)$$

with n the number density operator. It is inconvenient to calculate it directly. Instead, we introduce a corresponding temperature function that depends on the imaginary–time variables

$$\chi_n(\mathbf{x}_1\tau_1, \mathbf{x}_2\tau_2) = -\langle T_\tau[\tilde{n}(\mathbf{x}_1\tau_1)\tilde{n}(\mathbf{x}_2\tau_2)]\rangle \quad (A3)$$

where we have the following relation between Eqs. (A1) and (A3) with the Lehmann representation

$$X^R(\mathbf{k}, \omega) = \chi_n(\mathbf{k}, i\omega_n \to \omega + i0). \quad (A4)$$

The time–order indication T_τ of Eq. (A3) will disappear in the functional integral form,[10] which is the case in the present paper.

Next we notice that

$$
{}^{0}_{0}Q_{n_1 n_2} \cong \frac{i}{8}\sum_{\sigma}\left(\bar{\psi}_{n_1,\sigma}\psi_{n_2,\sigma} + \bar{\psi}_{n_2,\sigma}\psi_{n_1,\sigma}\right), \tag{A5a}
$$

$$
{}^{0}_{3}Q_{n_1 n_2} \cong \frac{1}{8}\sum_{\sigma}\left(\bar{\psi}_{n_1,\sigma}\psi_{n_2,\sigma} - \bar{\psi}_{n_2,\sigma}\psi_{n_1,\sigma}\right). \tag{A5b}
$$

By using Eqs. (A3) and (A5) we can then obtain

$$
\begin{aligned}
\chi_n(\mathbf{k},\omega_n) = 16T\sum_{1,2}\sum_{r=0,3}\Big\langle {}^{0}_{r}(\delta Q)_{1+n,1}(\mathbf{k}) \\
\times {}^{0}_{r}(\delta Q)_{2+n,2}(-\mathbf{k})\Big\rangle .
\end{aligned} \tag{A6}
$$

Similar analysis can be applied to find the spin density susceptibility. With the spin density

$$
\mathbf{n}_s(\mathbf{k},\omega_n) = \sqrt{\frac{T}{V}}\sum_{\mathbf{p},\omega}\big(\psi(\mathbf{p},\omega),\sigma\,\psi(\mathbf{p}+\mathbf{k},\omega+\omega_n)\big) \tag{A7}
$$

we can obtain Eq. (4.9).

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[1] See, e.g., G. Baym and C. Pethick, *Landau Fermi–Liquid Theory*, John Wiley (New York 1991).
[2] A. L. Fetter and D. L. Walecka, *Quantum Theory of Many–Particle Systems*, Mc Graw–Hill (New York 1971).
[3] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics*, Dover (New York 1975).
[4] R. Shankar, Rev. Mod. Phys. **66**, 129 (1994).
[5] Following Ref. 4, a large number of follow-ups, generalizations, and alternatives have been put forward, see Ref. 6 and references therein.
[6] D. Belitz and T.R. Kirkpatrick, Phys. Rev. B **56**, 6513 (1997); D. Belitz, F. Evers, and T.R. Kirkpatrick, Phys. Rev. B **58**, 9710 (1998); T.R. Kirkpatrick and D. Belitz, Phys. Rev. B **62**, 952 (2000); T.R. Kirkpatrick and D. Belitz, Phys. Rev. B **62**, 966 (2000).
[7] D. Belitz and T.R. Kirkpatrick, Phys. Rev. B **60**, 3485 (1999).
[8] L. Zhou, T.R. Kirkpatrick, unpublished.
[9] T.R. Kirkpatrick and D. Belitz, unpublished.
[10] See, e.g., J.W. Negele and H. Orland, *Quantum Many–Particle Systems*, Addison Wesley (New York 1988).
[11] For a pedagogical discussion of the replica trick, see, e.g., G. Grinstein, in *Fundamental Problems in Statistical Mechanics VI*, edited by E. G. D. Cohen, North Holland (Amsterdam 1985), p.147.
[12] T.R. Kirkpatrick and D. Belitz, Phys. Rev. Lett **66**, 1533 (1991); D. Belitz and T.R. Kirkpatrick, Phys. Rev. B **46**, 8393 (1992).
[13] A second contribution to $\mathcal{A}_{\rm dis}$, which is given in terms of $(\mathrm{tr}\,Q)^2$ rather than $\mathrm{tr}\,(Q^2)$,[6] is irrelevant for the purposes of this paper.
[14] D. Belitz and T.R. Kirkpatrick, Rev. Mod. Phys. **66**, 261 (1994).
[15] D. Belitz and T.R. Kirkpatrick, Phys. Rev. B **46**, 8393 (1992).
[16] P.W. Anderson, J. Phys. Chem. Solids **11**, 26 (1959).
[17] V. Dobrosavljevic, T.R. Kirkpatrick, Changfeng Chen and D. Belitz, Phys. Rev. B **44**, 5432 (1991).
[18] A.B. Pippard, Philos. Mag. **46**, 1104 (1955).
[19] D. Belitz, in *High Temperature Superconductivity*, edited by J.W. Lynn, Springer–Verlag (New York 1990), p.29.
[20] Moises Levy, Phys. Rev. **131**, 1497 (1963).
[21] G. Rickayzen, *Theory of Superconductivity*, John Wiley and Sons, Inc. (New York 1965), chap. 6.
[22] D. Pines and P. Nozières, *The Theory of Quantum Liquids*, Benjamin (New York, 1966), p. 206.
[23] M. Tinkham and R.A. Ferrell, Phys. Rev. Lett. **2**, 331 (1959); also see p. 218 of Ref. 22.
[24] M. Tinkham, *Introduction to superconductivity*, McGraw-Hill (New York, 1996).
[25] D.C. Mattis and J. Bardeen, Phys. Rev. **111**, 412 (1958); D. Belitz, S. De Souza-Machado, T.P. Devereaux and D.W. Hoard, Phys. Rev. B **39**, 2072 (1989).
[26] J.R. Schrieffer, *Theory of Superconductivity*, W. A. Benjamin (New York, 1964), p. 218.
[27] K. Yosida, Phys. Rev. **110**, 769 (1958).
[28] R.M. White, *Quantum theory of Magnetism*, Springer-Verlag (Berlin, 1983).
[29] T.P. Devereaux and D. Belitz, Phys. Rev. B **53**, 359 (1996).
[30] L. Zhou, Ph. D. Thesis, University of Maryland, unpublished.